ALE spaces from noncommutative U(1) instantons via exact Seiberg-Witten map

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ABSTRACT

The exact Seiberg-Witten (SW) map of a noncommutative (NC) gauge theory gives the commutative equivalent as an ordinary gauge theory coupled to a field dependent effective metric. We study instanton solutions of this commutative equivalent whose self-duality equation turns out to be the exact SW map of NC instantons. We derive general differential equations governing $U(1)$ instantons and we explicitly get an exact solution corresponding to the single NC instanton. Remarkably the effective metric induced by the single $U(1)$ instanton is related to the Eguchi-Hanson metric−the simplest gravitational instanton. Surprisingly the instanton number is not quantized but depends on an integration constant. Our result confirms the expected non-perturbative breakdown of the SW map. However, the breakdown of the map arises in a consistent way: The instanton number plays the role of a parameter giving rise to a one-parameter family of Eguchi-Hanson metrics.

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1 Introduction

Noncommutative (NC) spaces can be obtained by quantizing a given space with a symplectic structure $\theta_{\mu\nu}$:

$$[x^\mu, x^\nu]_\star = i\theta_{\mu\nu}, \tag{1.1}$$

where star-product is defined by

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\right) f(x)g(y)\Big|_{x=y}. \tag{1.2}$$

Also field theories can be formulated on the NC space. NC field theory means that fields are defined as functions over the NC space, whose products are defined by the NC star-product (1.2). At the algebraic level, the fields become operators acting on the Hilbert space as a representation space of Eq.(1.1).

On such space, the exponential $e^{ik\cdot x}$ acts as a translation operator, i.e.,

$$e^{ik\cdot x} \star f(x) \star e^{-ik\cdot x} = f(x + k \cdot \theta), \tag{1.3}$$

showing that a translation in a NC direction is equivalent to a gauge transformation up to a global symmetry transformation. This property is shared with another physical theory, General Relativity, where translations are also equivalent to gauge transformations. It is then natural to investigate the structure of general relativity inherent in the far simpler NC field theories. Our current work supports that NC gauge theories are really good toy models of general relativity, as asserted in [1].

To derive the action that governs NC gauge theories, we recall that they naturally arise as a decoupling limit of the open string dynamics on D-branes in the Neveu-Schwarz B field background. In the limit of slowly varying fields, the open string effective action on a D-brane is given by the Dirac-Born-Infeld (DBI) action [2]. Seiberg and Witten, however, showed [3] that an explicit form of the effective action depends on the regularization scheme of the two dimensional field theory defined by the worldsheet action. That is, depending on the regularization scheme or path integral prescription for the open string ending on a D-brane, one can have two descriptions: commutative and noncommutative descriptions. Since these two descriptions arise from the same open string theory and since the physics should not depend on the regularization scheme, it was argued in [3] that the two descriptions should be equivalent and thus there must be a spacetime field redefinition between ordinary and NC gauge fields, the so called Seiberg-Witten (SW) map. In this sense NC gauge theories have a dual description through the SW map in terms of ordinary gauge theories on commutative spacetime. To understand the dual description exactly, it is important to know the exact SW map.

If one uses the commutative description via the SW map, however, the connection between translation and gauge transformation is lost. A global translation on commutative fields can no longer be written as a gauge transformation. So one may wonder how the properties related to

gravity in NC gauge theories show up in the commutative description via the SW map. It turns out [4, 5] that, when the commutative description is employed, an "effective metric" induced by gauge fields directly emerges. Of course, this property suggests a possible gravitational interpretation. We will show that there exists a rigorous way of establishing this connection. More precisely, we will see that the effective metric generated by the single NC $U(1)$ instanton is related to the Eguchi-Hanson (EH) metric−the simplest gravitational instanton [6].

The paper is organized as follows. In Section 2, we briefly summarize the exact SW map obtained in [5, 7]. In Section 3, we derive the Bogomoln'yi bound for the commutative action obtained from the NC theory via the exact SW map. The resulting self-duality equation turns out to be the exact SW map of NC instantons. We derive the general differential equations governing $U(1)$ instantons and explicitly get an exact solution corresponding to a single NC instanton. We show that the instanton number is surprisingly not quantized but depends on an integration constant. This result shows a non-perturbative breakdown of the SW map whose possibility was already anticipated in [3] by Seiberg and Witten and in [8], more rigorously, by Harvey. In Section 4, we observe the remarkable fact that the single NC $U(1)$ instanton is mapped to the EH space. Furthermore, the breakdown of the SW map arises in a consistent way: The instanton number plays the role of a parameter giving rise to a one-parameter family of EH metrics. Since our effective metric is Kähler, we find the Kähler potential of the effective metric whose physical meaning from the gauge theory side is not yet obvious. Finally, in Section 5, we briefly summarize our results and discuss related open issues.

2 Exact Seiberg-Witten map of noncommutative gauge theory

The action for NC electrodynamics in flat Euclidean $\mathbf{R}^4$ is given by

$$\widehat{S}_{\rm NC} = \frac{1}{4} \int d^4x \, \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu}, \tag{2.1}$$

where noncommutative fields are defined by

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i\, [\widehat{A}_\mu, \widehat{A}_\nu]_\star. \tag{2.2}$$

For the reason mentioned in the Introduction, there should be a commutative deformed electrodynamics equivalent to Eq.(2.1). It was shown in [5, 7], for slowly varying fields on a single D-brane, that the dual description of the NC DBI action through the exact SW map is simply given by the ordinary DBI action expressed in terms of open string variables:

$$\int d^{p+1}x \sqrt{\det(G + \kappa(\widehat{F} + \Phi))}$$
$$= \int d^{p+1}x \sqrt{\det(1 + F\theta)} \sqrt{\det(G + \kappa(\Phi + \mathbf{F}))} + \mathcal{O}(\sqrt{\kappa}\partial F), \tag{2.3}$$

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where

$$\mathbf{F}_{\mu\nu}(x) \equiv \left(\frac{1}{1+F\theta}F\right)_{\mu\nu}(x) \tag{2.4}$$

and

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{2.5}$$

The commutative action in Eq.(2.3) is exactly the same as the DBI action obtained from the worldsheet sigma model using ζ-function regularization [9]. For a closed string background characterized by $B_{\mu\nu}, g_{\mu\nu}$ and g_s, we have a continuum of descriptions labelled by a choice of Φ. In the following, we are interested in $\Phi = 0$, the familiar NC description, the open string metric $G_{\mu\nu} = \delta_{\mu\nu}$ and $p = 3$.

In the zero slope limit $\kappa \equiv 2\pi\alpha' \to 0$, one can expand both sides of Eq.(2.3) in terms of powers of κ and produce infinitely many identities related to each other by the exact SW map. At $\mathcal{O}(\kappa^2)$, we get the exact commutative nonlinear electrodynamics equivalent to Eq.(2.1)

$$S_{\mathrm{C}} = \frac{1}{4} \int d^4x \sqrt{\det\mathrm{g}} \; \mathrm{g}^{\mu\alpha}\mathrm{g}^{\beta\nu} F_{\mu\nu}F_{\alpha\beta}, \tag{2.6}$$

where we introduced an "effective metric" induced by the dynamical gauge fields such that

$$\mathrm{g}_{\mu\nu} = \delta_{\mu\nu} + (F\theta)_{\mu\nu}, \qquad (\mathrm{g}^{-1})^{\mu\nu} \equiv \mathrm{g}^{\mu\nu} = \left(\frac{1}{1+F\theta}\right)^{\mu\nu}. \tag{2.7}$$

Note that the effective metric (2.7) is in general not symmetric. The action (2.6) is very interesting in the sense that the NC electrodynamics after the exact SW map can be regarded as the ordinary electrodynamics coupled to the "effective metric" $\mathrm{g}_{\mu\nu}$ [4]. It should be remarked, however, that the effective metric in the action (2.6) cannot be interpreted just as a fixed background since it depends on the dynamical gauge field. It is easy to derive the exact equation of motion from the action (2.6)

$$\partial_\mu\left[\sqrt{-\mathrm{g}}\left\{(\theta\mathrm{g}^{-1})^{\mu\alpha}\mathrm{Tr}\,(\mathrm{g}^{-1}F\mathrm{g}^{-1}F) - 2\left((\theta\mathrm{g}^{-1}F\mathrm{g}^{-1}F\mathrm{g}^{-1})^{\mu\alpha} - (\theta\mathrm{g}^{-1}F\mathrm{g}^{-1}F\mathrm{g}^{-1})^{\alpha\mu}\right)\right.\right.$$
$$\left.\left. +2\left((\mathrm{g}^{-1}F\mathrm{g}^{-1})^{\mu\alpha} - (\mathrm{g}^{-1}F\mathrm{g}^{-1})^{\alpha\mu}\right)\right\}\right] = 0. \tag{2.8}$$

It was checked in [5] that the nonlinear action Eq.(2.6) is consistent with the results in [10] where it was proved that the terms of order n in θ in the action via the SW map form a homogeneous polynomial of degree $n+2$ in F and explicitly presented the deformed action up to order θ^2.

The commutative action (2.6) can actually be also derived from the NC action (2.1) using the exact SW map in [7, 11, 12] (see [12, 13] for the exact inverse SW map):

$$\widehat{F}_{\mu\nu}(x) = \left(\frac{1}{1+F\theta}F\right)_{\mu\nu}(X), \tag{2.9}$$

$$d^4x = d^4X\sqrt{\det(1+F\theta)}(X), \tag{2.10}$$

where

$$X^\mu(x) \equiv x^\mu + \theta^{\mu\nu} \widehat{A}_\nu(x). \tag{2.11}$$

This is consistent with the above mentioned result by [10]. Here we used different coordinates, X^μ and x^μ, for commutative and NC descriptions, respectively. However we will often use the symbol x for both descriptions whenever the distinction is not necessary.

The exact SW map between topological invariants was also found in [7]. For example, using the SW maps (2.9) and (2.10) and the identity

$$\int d^4x \sqrt{\det g} \left(g^{-1}F \right) \wedge \left(g^{-1}F \right) = \int d^4x F \wedge F, \tag{2.12}$$

where the wedge notation has been used

$$F \wedge F \equiv \varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}, \tag{2.13}$$

one can get the exact SW map between instanton numbers

$$\int d^4x (\widehat{F} \wedge \widehat{F})(x) = \int d^4X (F \wedge F)(X). \tag{2.14}$$

The identity (2.12) will play a crucial role to derive the Bogomoln'yi bound of the action (2.6). The proof of the identity (2.12) is simple if one notices that the quantity $\left(g^{-1}F \right)_{\mu\nu}$ is anti-symmetric:

$$\sqrt{\det g} \left(g^{-1}F \right) \wedge \left(g^{-1}F \right) = 8\sqrt{\det g}\, \mathrm{Pf}(g^{-1}F) = 8\sqrt{\det g}\, \sqrt{\det(g^{-1}F)} = 8\mathrm{Pf}F = F \wedge F. \tag{2.15}$$

In next section, we will see that the identity (2.14) is interestingly broken, suggesting a non-perturbative breakdown of the SW map. We will further comment on this property of the map in Section 5.

3 Exact Seiberg-Witten map of noncommutative instantons

$\mathbf{R}^4$ is the simplest hyper-Kähler manifold, viewed as the quaternions $\mathbb{H} \simeq \mathbf{C^2}$. We introduce the quaternions $\mathbb{H}$ defined by

$$\mathbf{x} = x_\mu \sigma^\mu = \begin{pmatrix} x_4 + ix_3 & x_2 + ix_1 \\ -x_2 + ix_1 & x_4 - ix_3 \end{pmatrix} = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix} \tag{3.1}$$

$$\bar{\mathbf{x}} = x_\mu \bar{\sigma}^\mu = \begin{pmatrix} x_4 - ix_3 & -x_2 - ix_1 \\ x_2 - ix_1 & x_4 + ix_3 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix} \tag{3.2}$$

where $\sigma^\mu = (i\tau^a, 1)$ and $\bar{\sigma}^\mu = (-i\tau^a, 1) = -\sigma^2\sigma^{\mu T}\sigma^2$. The quaternion matrices σ^μ and $\bar{\sigma}^\mu$ have the basic properties

$$\sigma^\mu\bar{\sigma}^\nu = \delta^{\mu\nu} + i\sigma^{\mu\nu}, \qquad \sigma^{\mu\nu} = \eta^a_{\mu\nu}\tau^a = *\sigma^{\mu\nu},$$

$$\bar{\sigma}^\mu\sigma^\nu = \delta^{\mu\nu} + i\bar{\sigma}^{\mu\nu}, \qquad \bar{\sigma}^{\mu\nu} = \bar{\eta}^a_{\mu\nu}\tau^a = -*\bar{\sigma}^{\mu\nu}, \tag{3.3}$$

where the 4×4 matrices $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ are 't Hooft symbols defined by

$$\bar{\eta}^a_{ij} = \eta^a_{ij} = \varepsilon_{aij}, \qquad i, j \in \{1, 2, 3\},$$

$$\bar{\eta}^a_{4i} = \eta^a_{i4} = \delta_{ai}. \tag{3.4}$$

We list some identities of the 't Hooft tensors that will be useful for later calculations:

$$\eta^{(\pm)a}_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\lambda\kappa}\eta^{(\pm)a}_{\lambda\kappa}, \tag{3.5}$$

$$\eta^{(\pm)a}_{\mu\nu}\eta^{(\pm)a}_{\lambda\kappa} = \delta_{\mu\lambda}\delta_{\nu\kappa} - \delta_{\mu\kappa}\delta_{\nu\lambda} \pm \varepsilon_{\mu\nu\lambda\kappa}, \tag{3.6}$$

$$\eta^{(\pm)a}_{\mu\nu}\eta^{(\mp)b}_{\mu\nu} = 0, \tag{3.7}$$

$$\eta^{(\pm)a}_{\lambda\mu}\eta^{(\pm)b}_{\lambda\nu} = \delta_{ab}\delta_{\mu\nu} + \varepsilon_{abc}\eta^{(\pm)c}_{\mu\nu}, \tag{3.8}$$

where $\eta^{(+)a}_{\mu\nu} = \eta^a_{\mu\nu}$ and $\eta^{(-)a}_{\mu\nu} = \bar{\eta}^a_{\mu\nu}$.

The Euclidean Lorentz group $O(4)$ is isomorphic to $SU(2)_L \times SU(2)_R$, where two $SU(2)$ subgroups correspond to the left-handed and right-handed chiral rotations. The $O(4)$ group acts on the quaternion $\mathbf{x}$ as

$$\mathbf{x} \longrightarrow g_L\mathbf{x}g_R. \tag{3.9}$$

The self-dual (SD) and anti-self-dual (ASD) two-forms are basically given by the triple of Kähler 2-forms over $\mathbb{H}$

$$\omega^a_{SD} = -\frac{i}{4}\text{tr}_2(\tau^a d\mathbf{x} \wedge d\bar{\mathbf{x}}), \tag{3.10}$$

$$\omega^a_{ASD} = -\frac{i}{4}\text{tr}_2(\tau^a d\bar{\mathbf{x}} \wedge d\mathbf{x}), \tag{3.11}$$

where tr_2 denotes the trace over quaternionic indices. Note that only one part of Lorentz symmetry acts on the sphere of complex structures of the hyper-Kähler manifold $\mathbb{H} \simeq \mathbf{C^2}$:

$$\omega^a_{SD} \rightarrow -\frac{i}{4}\text{tr}_2(g^\dagger_L\tau^a g_L d\mathbf{x} \wedge d\bar{\mathbf{x}}), \tag{3.12}$$

$$\omega^a_{ASD} \rightarrow -\frac{i}{4}\text{tr}_2(g_R\tau^a g^\dagger_R d\bar{\mathbf{x}} \wedge d\mathbf{x}). \tag{3.13}$$

By the noncommutativity (1.1), the original Lorentz symmetry is broken down to its subgroup. For the SD and ASD $\theta_{\mu\nu}$, the original Lorentz symmetry $O(4) \cong SU(2)_L \times SU(2)_R$ is broken down to the subgroup

$$SU(2)_L \times SU(2)_R \rightarrow \begin{cases} SU(2)_R \times U(1)_L, & \text{SD,} \\ SU(2)_L \times U(1)_R, & \text{ASD.} \end{cases} \tag{3.14}$$

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Now we will restrict to the self-dual NC $\mathbf{R}^4$, with the canonical form $\theta_{\mu\nu} = \frac{\theta}{2}\eta_{\mu\nu}^3$. In this case the moduli space of the Lorentz symmetry breaking (3.14) is parameterized by $SU(2)_L/U(1)_L \cong \mathbf{S}^2$, which can be regarded as the Hopf map $\pi : \mathbf{S}^3 \to \mathbf{S}^2$ [14]. Using quaternions, the standard Hopf map can be represented as

$$T^a = -\frac{1}{4}\mathrm{tr}_2(\tau^3 \mathbf{x}\tau^a \bar{\mathbf{x}}).$$ (3.15)

In terms of $\mathbf{C}^2$ and $\mathbf{R}^4$ variables, they are explicitly given by

$$T^1 = -\frac{1}{2}(z_1\bar{z}_2 + \bar{z}_1 z_2) = -(x_1 x_3 + x_2 x_4),$$

$$T^2 = -\frac{i}{2}(z_1\bar{z}_2 - \bar{z}_1 z_2) = x_1 x_4 - x_2 x_3,$$ (3.16)

$$T^3 = \frac{1}{2}(z_1\bar{z}_1 - z_2\bar{z}_2) = \frac{1}{2}(x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

and

$$\sum_{a=1}^{3} T_a^2 = \frac{1}{4}r^4$$ (3.17)

with $r^2 = z_1\bar{z}_1 + z_2\bar{z}_2$. Under the Lorentz transformation (3.9), they transform as

$$T^a \to -\frac{1}{4}\mathrm{tr}_2(g_L^\dagger \tau^3 g_L \mathbf{x}(g_R\tau^a g_R^\dagger)\bar{\mathbf{x}}).$$ (3.18)

Since instanton solutions are the field configuration satisfying Bogomoln'yi bound, we also expect that the commutative instantons we want to find from the action (2.6) similarly satisfy the corresponding Bogomoln'yi bound of Eq.(2.6). Thus our problem is how to find the self-duality equation of Eq.(2.6) by applying the Bogomoln'yi trick. An essential hint comes from the fact that the left-hand side of Eq.(2.12) is a topological term defining instanton number on commutative $\mathbf{R}^4$. Guided by this fact, we rewrite the action (2.6) in the form

$$S_{\mathrm{C}} = \frac{1}{8}\int d^4x \sqrt{\mathrm{detg}}\Big(\mathbf{F}_{\mu\nu} \mp \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathbf{F}_{\alpha\beta}\Big)^2 \pm \frac{1}{8}\int d^4x \sqrt{\mathrm{detg}}\Big(\mathrm{g}^{-1}F\Big) \wedge \Big(\mathrm{g}^{-1}F\Big).$$ (3.19)

Note that the first term is positive definite since $d^4x\sqrt{\mathrm{detg}}$ is anyway the volume form of the NC coordinates (see Eq.(2.10)) and so positive definite, while the second term is topological because of Eq.(2.12) and thus does not affect the equations of motion. So we propose the self-duality equation for the action S_{C} to be

$$\mathbf{F}_{\mu\nu}(X) = \pm\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\mathbf{F}_{\alpha\beta}(X).$$ (3.20)

Note that the above equation is directly obtained by applying the exact SW map (2.9) to the NC self-duality equation, i.e.,

$$\widehat{F}_{\mu\nu}(x) = \pm\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\widehat{F}_{\alpha\beta}(x).$$ (3.21)

Using Eq.(2.12), one can check that the field configuration that satisfies the self-duality equation (3.20) also satisfies the equation of motion (2.8): Take a variation with respect to the gauge field A_μ on both sides of Eq.(2.12) and then the right-hand side identically vanishes. If the relation (3.20) is substituted to Eq.(2.8), the result is equal to the variation of the left-hand side of Eq.(2.12) with respect to A_μ and thus vanishes.

Since we want to consider a commutative equivalent of the Nekrasov-Schwarz instanton [15], we will consider the anti-self-dual case in Eq.(3.20). To solve the equation (3.20), we will take the following general strategy.

(I) Take a general ansatz with the ASD two-form basis ω_{ASD}^a as follows

$$\mathbf{F} \equiv \frac{1}{2}\mathbf{F}_{\mu\nu}dx^\mu \wedge dx^\nu = f^a(x)\omega_{ASD}^a, \tag{3.22}$$

where f^a's are arbitrary functions. Then the equation (3.20) is automatically satisfied.

(II) Solve the field strength $F_{\mu\nu}$ in terms of $\mathbf{F}_{\mu\nu}$:

$$F_{\mu\nu}(x) = \left(\frac{1}{1-\mathbf{F}\theta}\mathbf{F}\right)_{\mu\nu}(x). \tag{3.23}$$

Then impose the Bianchi identity for $F_{\mu\nu}$,

$$\varepsilon_{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \tag{3.24}$$

since the field strength $F_{\mu\nu}$ is given by a (locally) exact two-form, i.e., $F = dA$. In the end we will get general differential equations governing $U(1)$ instantons.

Substituting the ansatz (3.22), $\mathbf{F}_{\mu\nu}(x) = f^a(x)\bar{\eta}_{\mu\nu}^a$, into Eq.(3.23), we get

$$F_{\mu\nu} = \frac{1}{1-\phi}f^a\bar{\eta}_{\mu\nu}^a - \frac{2\phi}{\theta(1-\phi)}\eta_{\mu\nu}^3, \tag{3.25}$$

where

$$\phi \equiv \frac{\theta^2}{4}\sum_{a=1}^3 f^a(x)f^a(x). \tag{3.26}$$

Using the result (3.25) and Eq.(3.5), the Bianchi identity (3.24) is reduced to the following differential equations

$$f^a\bar{\eta}_{\mu\nu}^a\partial_\nu\phi + (1-\phi)\bar{\eta}_{\mu\nu}^a\partial_\nu f^a + \theta^{-1}\eta_{\mu\nu}^3\partial_\nu\phi^2 - \theta^{-1}\eta_{\mu\nu}^3\partial_\nu(1-\phi)^2 = 0. \tag{3.27}$$

From Eq.(3.25), we obtain

$$F_{\mu\nu}^+ \equiv \frac{1}{2}\left(F_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{4}(F\widetilde{F})\theta_{\mu\nu}^+ \tag{3.28}$$

since

$$F\widetilde{F} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -\frac{16\phi}{\theta^2(1-\phi)}. \tag{3.29}$$

Note that Eq.(3.28) is precisely the instanton equation, Eq.(4.54) in [3], used and explicitly solved there for single instanton case. See also [16]. Thus the instanton solution (4.60) in [3] was interestingly the exact solution although they got Eq.(3.28) perturbatively.

We will solve the differential equation (3.27) for the single instanton case. We will set $\theta = 1$ from now on, but it can be easily recovered by a simple dimensional analysis by recalling that θ carries the dimension of $(length)^2$. Since our instanton equation (3.20) (which is more fundamental than Eq.(3.28)) is obtained by the SW map from Eq.(3.21), we take an ansatz of the same form as the NC instanton [15, 17]

$$f^a(x) = f(r)T^a. \tag{3.30}$$

It is straightforward to derive an ordinary differential equation for the function $f(r)$ from Eq.(3.27):

$$r(r^2 f + 4)\frac{df}{dr} - 2(r^2 f - 12)f = 0, \tag{3.31}$$

where we assumed $(r^2 f + 4) \neq 0$ which turns out to be true. To get the result (3.31), the following relations might be useful

$$\bar{\eta}^a_{\mu\nu}\partial_\nu T^a = 3\eta^3_{\mu\nu}x^\nu, \qquad \bar{\eta}^a_{\mu\nu}x^\nu T^a = \frac{r^2}{2}\eta^3_{\mu\nu}x^\nu. \tag{3.32}$$

Eq.(3.31) is the well-known Abel's ordinary differential equation of the second kind. One may test asymptotic behaviors of the solution by assuming them as $f(r) = cr^{-n}$. The result is that $f(r) = 4/r^2$ when $r \to 0$ while $f(r) \sim c/r^6$ or cr^2 when $r \to \infty$. But we will require that the solution has to rapidly decay at $r \to \infty$ and thus we need $f(r) \sim c/r^6$ at $r \to \infty$. The exact solutions are given by

$$f(r) = \frac{4}{r^2}\frac{\sqrt{1 + \frac{t^4}{r^4}} \mp 1}{\sqrt{1 + \frac{t^4}{r^4}} \pm 1}, \tag{3.33}$$

where t^4 is an integration constant. Only the upper sign solution satisfies the correct asymptotic behavior.

The corresponding gauge field $A_\mu(x)$ can be found by taking the most general $SU(2)_R \times U(1)_L$ invariant ansatz [3]

$$A_\mu(x) = \eta^3_{\mu\nu}x^\nu h(r). \tag{3.34}$$

The expression about the field strength $F_{\mu\nu}$ can be found in Eq.(4.56) in [3]:

$$F_{12} = -2h - (x_1^2 + x_2^2)\frac{h'}{r}, \qquad F_{34} = -2h - (x_3^2 + x_4^2)\frac{h'}{r},$$

$$F_{13} = F_{24} = (x_1 x_4 - x_2 x_3)\frac{h'}{r}, \qquad F_{23} = -F_{14} = (x_2 x_4 + x_1 x_3)\frac{h'}{r}. \tag{3.35}$$

By comparing Eq.(3.35) with Eq.(3.25) with the ansatz (3.30), we can find the following relations

$$\frac{1}{1 - \frac{r^4}{16}f^2}f = -\frac{h'}{r}, \qquad \frac{r^4}{8(1 - \frac{r^4}{16}f^2)}f^2 = 2h + \frac{r}{2}h'. \tag{3.36}$$

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From these relations, we get

$$h(r) = \frac{r^2 f}{4 - r^2 f} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{t^4}{r^4}} \tag{3.37}$$

and, substituting Eq.(3.37) into Eq.(3.36), we get the differential equation (3.31) again. Conversely, the equation (3.31) is equivalent to

$$r\frac{d}{dr}(h^2 + h) = -4(h^2 + h). \tag{3.38}$$

Although the field strength $F_{\mu\nu}$ contains a (mild) singularity at $r = 0$, $\mathbf{F}_{\mu\nu}$ in Eq.(2.4) is completely non-singular. Since the mild singularity in $\sqrt{\det(1 + F\theta)} \sim t^2/4r^2$ for $r \to 0$ can be safely compensated by the volume term d^4x, the action (2.6) does not contain any harmful singularities. This behavior near $r = 0$ is exactly what is needed to give the solution a finite and nonzero instanton number. Let us calculate the instanton number:

$$I \equiv \frac{1}{32\pi^2} \int d^4x F\tilde{F} = -\frac{1}{16} \int_0^\infty dr \frac{r^7 f^2}{1 - \frac{1}{16}r^4 f^2} = -\frac{t^4}{16}. \tag{3.39}$$

Surprisingly the instanton number depends on the integration constant t^4. In a sense this fact was already shown in [3] since Seiberg and Witten solved exactly the same instanton equation (3.28) as ours. Since our approach used the exact SW map, i.e., included all corrections from θ-dependent terms, this result shows a non-perturbative breakdown of the SW map. However, this possibility of the map was already anticipated in [3, 18] and, more rigorously, in [8], where it was pointed out that commutative and NC gauge fields have different topology. Indeed there is no topological reason that commutative $U(1)$ instantons have a quantized topological charge.

The behavior in the commutative limit, i.e., $\theta \to 0$, can be clarified by recovering the dimensionful parameter θ. In particular, one has to set $t^2 = \theta\tilde{t}^2$ with a dimensionless constant $\tilde{t}$. In the limit of small θ and fixed $\tilde{t}$, the solution (3.33) goes to zero, while the quantity I in Eq.(3.39) remains constant. In this way, one does not recover the strictly commutative case. We notice, however, that fixing the quantity $\theta\tilde{t}^4$ and sending θ to zero gives $f(r) \sim \theta\tilde{t}^4/r^6$ (an additional θ^{-1} appears in front of Eq.(3.33) after dimensional analysis). The Nekrasov-Schwarz instanton [15] also exhibits the same behavior in the $\theta \to 0$ limit. This behavior, $f(r) \sim C/r^6$, is also proper of the solution in the strictly commutative case, with a fixed dimensionful constant C. Taking the commutative limit, we thus get $t^2 \to 0$ while $\tilde{t}^4 \to \infty$. We recover in this way the result for the solution of the strictly linear equation $F_{\mu\nu}^+ = 0$ with the same symmetry [3] where $I \propto \tilde{t}^4 \to \infty$.

4   ALE spaces from $U(1)$ instantons

In the Introduction, we speculated about the possibility that NC gauge fields can play the role of gravity. In this section, we will try to give a precise mathematical and physical connection

between $U(1)$ instantons on $\mathbf{R}^4$ with flat metric and gravitational instantons, which are hyper-Kähler four-manifolds arising in general relativity.

Let us rewrite the "effective metric" in Eq.(2.7) in the form [1]

$$g_{\mu\nu} = \frac{1}{2}(\delta_{\mu\nu} + \widetilde{g}_{\mu\nu}). \tag{4.1}$$

It is straightforward to get the metric $\widetilde{g}_{\mu\nu}$ using Eq.(3.35) with the solution (3.37):

$$\widetilde{g}_{\mu\nu} = \begin{pmatrix} G_1 & 0 & G_4 & G_3 \\ 0 & G_1 & -G_3 & G_4 \\ G_4 & -G_3 & G_2 & 0 \\ G_3 & G_4 & 0 & G_2 \end{pmatrix}, \tag{4.2}$$

where

$$G_1 = G - H(x_1^2 + x_2^2), \qquad G_2 = G - H(x_3^2 + x_4^2),$$
$$G_3 = -H(x_1 x_4 - x_2 x_3), \qquad G_4 = -H(x_2 x_4 + x_1 x_3)$$

and

$$G = \frac{\sqrt{r^4 + t^4}}{r^2}, \qquad H = -\frac{G'}{2r} = \frac{t^4}{r^4\sqrt{r^4 + t^4}}.$$

First we note that $\widetilde{g}_{\mu\nu} = \delta_{\mu\nu} + \mathcal{O}(r^{-4})$ at infinity, which is a common property of a particular family of hyper-Kähler manifold, the so-called ALE spaces [19, 20]. Indeed this metric is precisely the EH metric, the simplest ALE space,[2] in the form that can be found in [21]. We thus constructed the gravitational instanton from $U(1)$ gauge fields in the *flat* spacetime. The integration constant t^4 plays the role of a parameter giving rise to a one-parameter family of EH metrics. This metric becomes flat in the case of $t = 0$ (except at the origin). Thus the family of the EH space is parameterized by the instanton number !

The instanton number is now endowed with a meaning by the Kähler geometry. So the $U(1)$ instantons are consistently connected to the hyper-Kähler geometries. In this way, commutative $U(1)$ instantons behave as a source that generates Kähler geometries. If one insists on the (anti-)self-dual configurations of the action (2.6), it thus looks like a theory of Kähler geometry rather than a theory of gauge fields.

Since our effective metric, $ds^2 = \widetilde{g}_{i\bar{j}} dz_i d\bar{z}_j$, $i, j = 1, 2$, is hyper-Kähler, we can introduce a Kähler two-form Ω and a Kähler potential K defined by

$$\Omega = \frac{i}{2}\widetilde{g}_{i\bar{j}} dz_i \wedge d\bar{z}_j = \frac{i}{2}\partial\bar{\partial}K \tag{4.3}$$

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[1]Since the effective metric $g_{\mu\nu} = (1 - \mathbf{F}\theta)^{-1}_{\mu\nu}$, the metric determined by Eq.(3.20) is symmetric if $\varepsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\mu\nu}\theta_{\rho\sigma} = 0$.

[2]It may be worthwhile to point out that the isometry group of the instanton and the EH space is also coincident with $SU(2)_R \times U(1)_L$.

where the exterior derivative is defined by

$$d = dx^\mu \frac{\partial}{\partial x^\mu} = dz^i \partial_i + d\bar{z}^i \bar{\partial}_i = \partial + \bar{\partial}. \tag{4.4}$$

Using the definition (4.3), one can easily check that the Kähler potential K [22, 20, 21] for the EH metric (4.2) is given by

$$K = \sqrt{r^4 + t^4} + t^2 \log \frac{r^2}{\sqrt{r^4 + t^4} + t^2}. \tag{4.5}$$

It is very remarkable for the $U(1)$ instanton to reproduce precisely the Kähler potential for the EH metric. However, it is not yet obvious what is the physical meaning of the Kähler potential from the gauge theory side although it is an important ingredient in the Kähler geometry. We leave this interpretation for future work.

5   Discussion

We studied the commutative instantons related to NC instantons by the exact SW map.[3] We found self-duality equations, from which we got general differential equations governing $U(1)$ instantons. We observed that our self-duality equation is equivalent to the instanton equation in [3]. We also found that the instanton number is no longer quantized. This result suggests a non-perturbative breakdown of the SW map. Let us further discuss this last property.

The SW map is a map between gauge orbit spaces of commutative and NC gauge fields. However, it was pointed out [3, 18] that the change of variables from $\widehat{A}_\mu$ to A_μ or vice versa has only a finite radius of convergence. Thus the SW map cannot completely encode the topology of gauge fields. Indeed it was shown [8] that the gauge orbit spaces for commutative and noncommutative gauge theories are different. In particular, the topology of NC $U(1)$ gauge fields is nontrivial while the commutative one is trivial. Our result confirms these differences. The other example of this breakdown is the level quantization of NC Chern-Simons theory for $U(1)$ gauge group [25]. (For the exact SW map of the Chern-Simons theory, see [26].) However this seems to be interrelated to the instanton case due to the following relation [7]

$$\int d^4 x \widehat{F} \wedge \widehat{F} = \int d^4 x \, d\widehat{\Omega}_{\mathrm{CS}} \tag{5.1}$$

with

$$\widehat{\Omega}_{\mathrm{CS}} = \int_0^1 dt \widehat{A} \wedge \widehat{F}_t \tag{5.2}$$

where $\widehat{F}_t = td\widehat{A} - it^2 \widehat{A} \star \widehat{A}$. We derived the identity (2.14) from the exact SW map. But we observed that the identity (2.14) is broken down: The left-hand side carries a nontrivial topology

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[3]The SW map of NC instantons was previously studied in [23] for localized intantons generated by shift operators and in [24] for the Nekrasov-Schwarz instantons.

while the right-hand side carries trivial one. So, if one introduces a three-manifold $\mathcal{M}_3$ such that $\partial\mathcal{M}_4 = \mathcal{M}_3$, one gets a different character of level quantization for the commutative and NC Chern-Simons theories.

We showed that our effective metric induced by the commutative $U(1)$ instanton is interestingly related to a particular family of hyper-Kähler manifold, the so-called ALE spaces, at least for the simplest instanton. We thus constructed the gravitational instanton from $U(1)$ gauge fields in *flat* spacetime. Note that there is a general construction of all ALE manifolds by Kronheimer [19]. In this construction ALE spaces are explicitly obtained as hyper-Kähler quotients of flat Euclidean spaces [27] and emerge as minimal resolutions of $\mathbf{C}^2/\Gamma$, where Γ is a discrete subgroup of $SU(2)$. For ALE spaces of A-type, corresponding to $\Gamma = \mathbb{Z}_N$, the metric is known to be diffeomorphic to the Gibbons-Hawking multi-center metric [28]. (The two-center Gibbons-Hawking metric is the EH metric and the hyper-Kähler quotient construction of the EH metric is given in the Appendix in [29].) So it is a very interesting problem to explore whether more general ALE spaces can be obtained by $U(1)$ instantons in the way we examined in this paper. Note that the EH metric was obtained by the very special ansatz (3.30) which is the Hopf map with unit Hopf invariant. One may try an ansatz described by a Hopf map with higher Hopf invariants. Conversely, one may try to find an instanton solution corresponding to, for example, the multi-center Gibbons-Hawking metric.

Another interesting problem is how to embed the commutative $U(1)$ instantons described by Eq.(3.20) into the ADHM construction. At first sight, this seems not possible since the instanton number is not quantized. However, if the instanton solution of Eq.(3.20) is generally related to the ALE spaces, one may rather extract the instanton solutions from the Kronheimer's hyper-Kähler quotient construction of the ALE spaces [19]. In this construction, the instanton number appears as a deformation parameter in the moment map $\mu_{\mathbf{R}}$; $\mu_{\mathbf{R}} = t^2$ [29]. Setting the deformation parameter $t^2 = 0$, we get the singular orbifold $\mathbf{C}^2/\Gamma$, consistent with the metric (4.2). The commutative limit discussed at the end of Section 3 thus corresponds to the singlular limit of ALE spaces, as naturally expected result.

ALE spaces carry two topological invariants: the Euler characteristic and the Hirzebruch signature [30]. A natural question arising from our work is what is the meaning of these topological invariants from the gauge theory point of view. Since they should be represented by higher derivative terms of $U(1)$ field strength $F_{\mu\nu}$, they are very exotic objects in the gauge theory picture. We hope to address some of the problems raised here in the near future.

Acknowledgments

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